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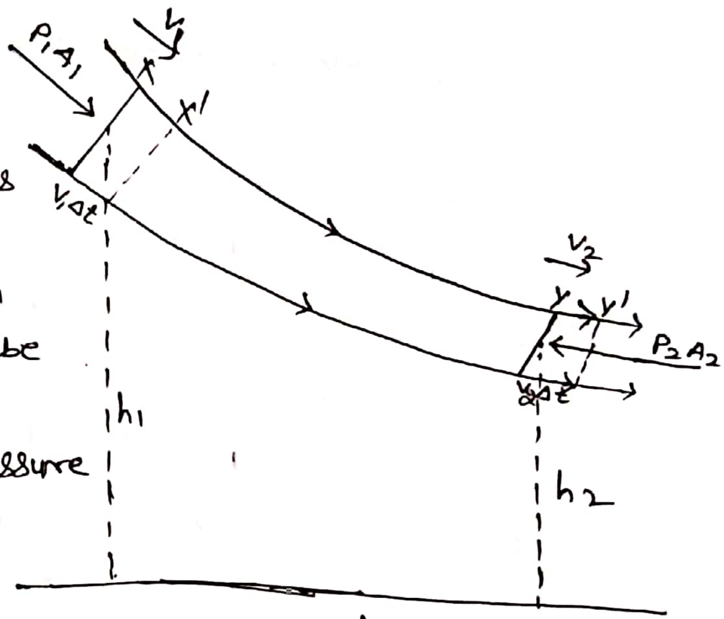
Bernoulli's Theorem :- It state that for all points along a streamline in a incompressible and non-viscous fluid flowing steadily, the sum of the pressure energy, the potential energy per unit volume and the kinetic energy per unit volume is constant.

Proof :-



Let an incompressible and non-viscous fluid flowing steadily, through a non-uniform tube of flow. Let P_1 and v_1 be the pressure and velocity at a cross section x and P_2 and v_2 the pressure and velocity at another cross section y. Let A_1 be the area of the cross section x and h_1 its mean height above an arbitrary horizontal level, and similarly, let A_2 be the area and h_2 the mean height of the cross-section y above the same level. If the area A_2 is ~~Sma~~ smaller then A_1, then by the equation of continuity, the velocity v_2 is greater then v_1.

Let us consider the flow of the liquid included between x and y. At x it is acted upon by a pressure force P_1A_1 in the direction of flow exerted by the fluid to its left, and y it is acted upon by a pressure force P_2A_2 in the direction opposite to the flow exerted by the fluid to its rights.

In a small time interval Δt, the left end of the fluid xy advances from x to x' through a distance $v_1 \Delta t$ parallel to the force $P_1 A_1$. Therefore the work done on the end xy by the force $P_1 A_1$ is

$$(P_1 A_1)(v_1 \Delta t)$$

In the same time interval, the right end of the fluid xy advances from y to y' through distance $v_2 \Delta t$ against the force $P_2 A_2$. Therefore the work done by the fluid xy against the force is

$$(P_2 A_2)(v_2 \Delta t)$$

Hence the net work done by the pressure force on the fluid is

$$P_1 A_2 v_1 \Delta t - P_2 A_2 v_2 \Delta t$$

But $A_1 v_1 \Delta t$ and $A_2 v_2 \Delta t$ are the volumes of the fluid that crossed x and respectively during the interval Δt. i.e. volumes must be equal because the fluid is incompressible. If Δm be the mass of either volume and ρ the density of the fluid we, have

$$A_1 v_1 \Delta t = A_2 v_2 \Delta t = \frac{\Delta m}{\rho}$$

and finally, net work done on the fluid

$$xy = \cancel{(P_1 - P_2)\frac{\Delta m}{\rho}}$$

$$= P_1 \cdot \frac{\Delta m}{\rho} - P_2 \frac{\Delta m}{\rho}$$

$$= (P_1 - P_2)\frac{\Delta m}{\rho}$$

Since the fluid is non-viscous, the net work done on it must be equal to the net gain in its mechanical energy as it flows from xy and x'y'.

In the New position the potential energy (P.E) of the fluid has decreased by the difference between the P.E. of mass Δm of the fluid which was orginally between x and x' and the P.E of the same mass Δm of the fluid between y and y'. This is clearly, equal to

$$(\Delta m)gh_1 - (\Delta m)gh_2 = (\Delta m)g(h_1-h_2)$$

Similarly, the increase in kinetic Energy (K.E) of the fluid as it moves from xy to x'y' is equal to the K.E of the fluid which appears in yy' minus the K.E which disappears from xx'. This is equal to

$$\frac{1}{2}(\Delta m)v_2^2 - \frac{1}{2}(\Delta m)v_1^2 = \frac{1}{2}(\Delta m)(v_2^2-v_1)^2$$

Hence the net gain in the mechanical energy is equal to

$$\frac{1}{2}(\Delta m)(v_2^2-v_1^2) - (\Delta m)g(h_1-h_2)$$

Equating the net work done on the fluid to the net gain in its mechanical energy

$$(P_1-P_2)\frac{\Delta m}{\rho} = \frac{1}{2}\Delta m(v_2^2-v_1^2) - (\Delta m)g(h_1-h_2)$$

Dividing by Δm and multiplying by ρ we get

$$(P_1-P_2) = \frac{1}{2}\rho(v_2^2-v_1^2) - \rho g(h_1-h_2)$$

Rearranging the terms,

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 - \rho g h_1 + \rho g h_2$$

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

Thus along a streamline the quantity

$$\boxed{P + \rho g h + \frac{1}{2}\rho v^2 = \text{Constant}}$$

This is Called Bernoulli's equation for steady; non-viscous; Incompressible flow.

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

$$\text{or,} \quad P + \frac{1}{2}\rho v^2 = \text{Constant}$$

This shows that where the velocity of flow is less, the pressure is larger and vice-versa.